1.a. Para el conjunto de datos $\boldsymbol{x} = \{x_1, x_2, \dots, x_{10}\}$. Tenemos,

$$\sum_{i=1}^{10} x_i = 140, \qquad \sum_{i=1}^{10} x_i^2 = 5230.$$

De este modo, $\overline{x} = 140/10 = 14$. Mientras que,

$$\begin{aligned} s^2 &= \frac{1}{10-1}\sum_{i=1}^{10}(x_i - \overline{x})^2 = \frac{1}{9}\Big(\sum_{i=1}^{n} x_i^2 - 10\cdot\overline{x}^2\Big) = \frac{1}{9}\big(5230 - 10\cdot 14^2\big) \\ &= \frac{1}{9}(5230 - 1960) = \frac{3270}{9} = \frac{1090}{3} = 363.3333. \end{aligned}$$

Además,

$$\mathsf{CV} = \frac{s}{\overline{x}} = \frac{\sqrt{1090/3}}{14} = \frac{19.0613}{14} = 1.3615.$$

1.b. Tenemos que los valores ordenados, $x_{(1)} < x_{(2)} < \cdots < x_{(10)}$ son dados por:

$$2,\ 3,\ 5,\ 7,\ 8, 10, 11, 12, 15, 67.$$

Como $n = 10$, sigue que

$$\mathsf{me} = \frac{8+10}{2} = 9.$$

Para calcular Q_1 y Q_3 considere los nuevos conjuntos de datos ordenados

$$\boldsymbol{D}_1 = \{2, 3, 5, 7, 8\}, \quad \text{y} \quad \boldsymbol{D}_2 = \{10, 11, 12, 15, 67\}.$$

De ahí que $Q_1 = 5$ y $Q_3 = 12$, y $IQR = Q_3 - Q_1 = 12 - 5 = 7$. Esto permite obtener

$$b_{\mathsf{G}} = \frac{(Q_3 - \mathsf{me}) - (\mathsf{me} - Q_1)}{IQR} = \frac{(12-9)-(9-5)}{7} = \frac{3-4}{7} = -\frac{1}{7} = -0.1429.$$

1.c. Sabemos que

$$\begin{aligned} \overline{y} &= -1.3\overline{x} + 7.1 = -1.3\cdot 14 + 7.1 = -11.1 \\ \mathsf{var}(\boldsymbol{y}) &= (-1.3)^2\,\mathsf{var}(\boldsymbol{x}) = 1.69\cdot 363.3333 = 614.0333. \end{aligned}$$

De este modo,

$$\mathsf{CV}_y = \frac{\sqrt{\mathsf{var}(\boldsymbol{y})}}{|\overline{y}|} = \frac{\sqrt{614.0333}}{11.2} = \frac{24.7797}{11.2} = 2.2125.$$

1.d. Note que

$$y_i = ax_i + b, \qquad i = 1, 2, \dots, n,$$

con $a = -1.3$ y $b = 7$. Esto nos permite escribir

$$\begin{aligned}\operatorname{cov}(\boldsymbol{x}, \boldsymbol{y}) &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(ax_i + b - a\overline{x} - b)\\ &= a \cdot \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 = a \operatorname{var}(\boldsymbol{x}).\end{aligned}$$

Es decir, $\operatorname{cov}(\boldsymbol{x}, \boldsymbol{y}) = (-1.3) \cdot 1090/3 = -472.3333$. Como $\operatorname{var}(\boldsymbol{y}) = a^2 \operatorname{var}(\boldsymbol{x})$, sigue que

$$\operatorname{corr}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\operatorname{cov}(\boldsymbol{x}, \boldsymbol{y})}{\sqrt{\operatorname{var}(\boldsymbol{x}) \operatorname{var}(\boldsymbol{y})}} = \frac{a \operatorname{var}(\boldsymbol{x})}{\sqrt{a^2 \operatorname{var}^2(\boldsymbol{x})}},$$

y como en nuestro caso particular $a < 0$, sigue que

$$\operatorname{corr}(\boldsymbol{x}, \boldsymbol{y}) = \frac{a}{\sqrt{a^2}} = \frac{a}{|a|} = -1.$$

2. Desarrollando el cuadrado de binomio y sumando, obtenemos

$$\begin{aligned}\sum_{i=1}^{k} n_i(x_i - \overline{x})^2 &= \sum_{i=1}^{k} n_i(x_i^2 - 2x_i\overline{x} + \overline{x}^2) = \sum_{i=1}^{k} n_i x_i^2 - 2\overline{x}\sum_{i=1}^{k} n_i x_i + \overline{x}^2 \sum_{i=1}^{k} n_i\\ &= \sum_{i=1}^{k} n_i x_i^2 - 2n\overline{x}^2 + n\overline{x}^2 = \sum_{i=1}^{k} n_i x_i^2 - n\overline{x}^2,\end{aligned}$$

lo que verifica el resultado.

3. Tenemos $n = 6$, y

$$\begin{aligned}&\sum_{i=1}^{n} x_i = 21, \qquad \sum_{i=1}^{n} x_i^2 = 91, \qquad \sum_{i=1}^{n} x_i y_i = 280\\ &\sum_{i=1}^{n} y_i = 65, \qquad \sum_{i=1}^{n} y_i^2 = 879.\end{aligned}$$

Es decir,

$$\begin{aligned}&\sum_{i=1}^{n}(x_i - \overline{x})^2 = \sum_{i=1}^{n} x_i^2 - n\overline{x}^2 = 91 - 21^2/6 = 17.5000\\ &\sum_{i=1}^{n}(y_i - \overline{y})^2 = \sum_{i=1}^{n} y_i^2 - n\overline{y}^2 = 879 - 65^2/6 = 174.8333\\ &\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = \sum_{i=1}^{n} x_i y_i - n\overline{x}\,\overline{y} = 280 - 21 \cdot 65/6 = 52.5000.\end{aligned}$$

De este modo,

$$\widehat{\beta} = \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{52.5}{17.5} = 3.0,$$

y portanto $\widehat{\alpha} = \overline{y} - \widehat{\beta}\overline{x} = (65 - 3 \cdot 21)/6 = 1/3$. Además, tenemos que los residuos, $e_i = y_i - \widehat{\alpha} - \widehat{\beta} x_i$, para $i = 1, \dots, n$, son dados por:

$$\boldsymbol{e} = \{5/3, 2/3, -7/3, -7/3, 2/3, 5/3\},$$

De este modo,

$$RSS = \sum_{i=1}^n e_i^2 = \frac{1}{3^2}\big(5^2 + 2^2 + (-7)^2 + (-7)^2 + 2^2 + 5^2\big) = \frac{156}{9} = 17.3333.$$

Es decir, $s^2 = RSS/(n-2) = 17.3333/4 = 4.3333$. Finalmente

$$R^2 = 1 - \frac{RSS}{\sum_{i=1}^n (y_i - \overline{y})^2} = 1 - \frac{17.3333}{174.3333} = 0.9009.$$

4.a. Tenemos que

$$\frac{\mathsf{d}}{\mathsf{d}\,\theta} S_1(\theta) = \sum_{i=1}^n \frac{\mathsf{d}}{\mathsf{d}\,\theta}(y_i - \theta x_i)^2 = 2\sum_{i=1}^n (y_i - \theta x_i)\frac{\mathsf{d}}{\mathsf{d}\,\theta}(y_i - \theta x_i)$$
$$= -2\sum_{i=1}^n (y_i - \theta x_i) x_i.$$

Resolviendo la condición $\mathsf{d}\, S_1(\theta)/\,\mathsf{d}\,\theta = 0$, sique que

$$\sum_{i=1}^n (y_i - \theta x_i) x_i = 0 \qquad \Longrightarrow \qquad \widehat{\theta} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}.$$

Notando que

$$\frac{\mathsf{d}^2}{\mathsf{d}\,\theta^2} S_1(\theta) = 2\sum_{i=1}^n x_i^2 > 0,$$

para cualquier $\theta \in \mathbb{R}$, sigue que $\widehat{\theta}$ es mínimo global.

4.b. Para los datos del Ejercicio 3, tenemos

$$\widehat{\theta} = \frac{280}{91} = 3.0769.$$

Calculando $e_i = y_i - \widehat{\theta} x_i$, para $i = 1, \dots, 6$, resulta

$$\boldsymbol{e} = \{1.9231, 0.8462, -2.2308, -2.3077, 0.6154, 1.5385\}.$$

De este modo,

$$s_*^2 = \frac{1}{6-1}\sum_{i=1}^6 e_i^2 = \frac{17.4620}{5} = 3.4924.$$